arXiv:math/0003066v2 [math.QA] 10 Jul 2000

GENERALIZED JORDANIAN R-MATRICES OF CREMMER-GERVAIS TYPE

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Abstract. An explicit quantization is given of certain skew-symmetric solutions of the classical Yang-Baxter equation, yielding a family of R-matrices which generalize to higher dimensions the Jordanian R-matrices. Three different approaches to their construction are given: as twists of degenerations of the Shibukawa-Ueno Yang-Baxter operators on meromorphic functions; as boundary solutions of the quantum Yang-Baxter equation; via a vertex-IRF transformation from solutions to the dynamical Yang-Baxter equation.

Introduction

Let $\mathbb{F}$ be an algebraically closed field of characteristic zero. The skew-symmetric solutions of the classical Yang-Baxter equation for a simple Lie algebra are classified by the quasi-Frobenius subalgebras; that is, pairs of the form $(\mathfrak{f},\omega)$ where $\mathfrak{f}$ is a subalgebra and $\omega : \mathfrak{f}\wedge\mathfrak{f}\to\mathbb{F}$ is a nondegenerate 2-cocycle on $\mathfrak{f}$. By a result of Drinfeld [6], the associated Lie bialgebras admit quantizations. This is done by twisting the enveloping algebra $U(\mathfrak{g})[[h]]$ by an appropriate Hopf algebra 2-cocycle. However neither construction lends itself easily to direct calculation and few explicit examples exist to illustrate this theory. The most well-known is the Jordanian quantum group [4, 16] associated to the classical r-matrix $E\wedge H$ inside $\mathfrak{sl}(2)\otimes\mathfrak{sl}(2)$. In [12], Gerstenhaber and Giaquinto constructed explicitly the r-matrix $r_{\mathfrak{p}}$ associated to certain maximal parabolic subalgebras $\mathfrak{p}$ of $\mathfrak{sl}(n)$. In particular for the parabolic subalgebra $\mathfrak{p}$ generated by $\mathfrak{b}^+$ and $F_1,\dots F_{n-2}$, their construction yields

$$r_{\mathfrak{p}} = n\sum_{i<j}\sum_{k=i}^{j-1} E_{k,i}\wedge E_{i+j-k-1,j} + \sum_{i,j}(j-1)E_{j-1,j}\wedge E_{i,i}$$

In [13], they raise the problem of quantizing this r-matrix, in the sense of constructing an invertible $R\in M_n(\mathbb{F})\otimes M_n(\mathbb{F})\otimes\mathbb{F}[[h]]$ satisfying the Yang-Baxter equation and of the form $I+hr+O(h^2)$. When $n=2$, the solution is the well-known Jordanian R-matrix. Gerstenhaber and Giaquinto construct a quantization of $r_{\mathfrak{p}}$ in the $n=3$ case and verify the necessary relations by direct calculation. We give below the quantization of $r_{\mathfrak{p}}$ in the general case. Moreover, we are able to give three separate constructions which emphasize the fundamental position occupied by this R-matrix.

In the first section we construct R (somewhat indirectly) as an extreme degeneration of the Belavin R-matrix. We do this by following the construction by

Date: June 9, 2000.

The second author was supported in part by NSA grant MDA904-99-1-0026 and by the Charles P. Taft Foundation.

Shibukawa and Ueno of solutions of the Yang-Baxter equation for linear operators on meromorphic functions. In [17], they showed that from any solution of Riemann's three-term equation, they could construct such a solution of the Yang-Baxter equation. These solutions occur in three types: elliptic, trigonometric and rational. Felder and Pasquier [11] showed that in the elliptic case, these operators, after twisting and restricting to suitable finite dimensional subspaces, yield Belavin's R-matrices. In the trigonometric case, the same procedure yields the affinization of the Cremmer-Gervais quantum groups; sending the spectral parameter to infinity then yields the Cremmer-Gervais R-matrices themselves. Repeating this procedure in the rational case yields the desired quantization of $r_{\mathfrak{p}}$, which we shall denote $R_{\mathfrak{p}}$.

In the second section we show that these R-matrices occur as boundary solutions of the modified quantum Yang-Baxter equation, in the sense of Gerstenhaber and Giaquinto [13]. It was observed in [12] that if $\mathfrak{M}$ is the set of solutions of the modified classical Yang-Baxter equation, then $\mathfrak{M}$ is a locally closed subset of $\mathbb{P}(\mathfrak{g} \wedge \mathfrak{g})$ and $\bar{\mathfrak{M}} - \mathfrak{M}$ consists of solutions to the classical Yang-Baxter equation. The element $r_{\mathfrak{p}}$ was found to lie on the boundary of the orbit under the adjoint action of $SL(n)$ of the modified Cremmer-Gervais r-matrix. In [13], Gerstenhaber and Giaquinto began an investigation into the analogous notion of boundary solutions of the quantum Yang-Baxter equation. They conjectured that the boundary solutions to the classical Yang-Baxter equation described above should admit quantizations which would be on the boundary of the solutions of their modified quantum Yang-Baxter equation. They confirmed this conjecture for the Cremmer-Gervais r-matrix in the $\mathfrak{sl}(3)$ case using some explicit calculations. We prove the conjecture for the general Cremmer-Gervais r-matrix by verifying that the matrices $R_{\mathfrak{p}}$ do indeed lie on the boundary of the set of solutions to the modified quantum Yang-Baxter equation.

In the third section we show that these matrices may also be constructed via a "Vertex-IRF" transformation from certain solutions of the dynamical Yang-Baxter equation given in [7]. This construction is analogous to the original construction of the Cremmer-Gervais R-matrices given in [3].

The position of $R_{\mathfrak{p}}$ with relation to other fundamental solutions of the YBE and DYBE can be summarized heuristically by the diagram below.

$$\begin{array}{ccccccc} R_B & & & & R_F & & \\ \downarrow & & & & \downarrow & & \\ \hat{R}_{CG} & \longrightarrow & R_{CG} & & \hat{R}_{GN} & \longrightarrow & R_{GN} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ R_{B,r} & \longrightarrow & R_{\mathfrak{p}} & & R_{F,r} & \longrightarrow & R_{GN,r} \\ & \text{YBE} & & & & \text{DYBE} & \end{array}$$

On the left hand side, R_B is Belavin's elliptic R-matrix; R_{CG} the Cremmer-Gervais R-matrix; $\hat{R}_{CG}$ is the affinization of R_{CG} which is also the trigonometric degeneration of the Belavin R-matrix; $R_{B,r}$ is a rational degeneration of the Belavin R-matrix. The vertical arrows denote degeneration of the coefficient functions (from elliptic to trigonometric and from trigonometric to linear); the horizontal arrows denote the limit as the spectral parameter tends to infinity. On the right hand side,

R_F is Felder's elliptic dynamical R-matrix; $\hat{R}_{GN}$ and $R_{F,r}$ are trigonometric and rational degenerations; R_{GN} is the Gervais-Neveu dynamical R-matrix and $R_{GN,r}$ is a rational degeneration of the Gervais-Neveu matrix given in [7]. The passage between the two diagrams is performed by Vertex-IRF transformations. The relationships involved in the top two lines of this diagram are well-known [1, 3, 10]. This paper is concerned with elucidating the position of $R_{\mathfrak{p}}$ in this picture.

The authors would like to thank Tony Giaquinto for many helpful conversations concerning boundary solutions of the Yang-Baxter equation.

1. Construction of $R_{\mathfrak{p}}$

1.1. **The YBE for operators on function fields.** Recall that if A is an integral domain and σ is an automorphism of A, then σ extends naturally to the field of rational functions $A(x)$ by acting on the coefficients. Denote by $\mathbb{F}(z_1, z_2)$ the field of rational functions in the variables z_1 and z_2. Then for any $\sigma \in \operatorname{Aut} \mathbb{F}(z_1, z_2)$, and any $i, j \in \{1, 2, 3\}$, we may define $\sigma_{ij} \in \operatorname{Aut} \mathbb{F}(z_1, z_2, z_3)$ by realizing $\mathbb{F}(z_1, z_2, z_3)$ as $\mathbb{F}(z_i, z_j)(z_k)$. Set $\Gamma = \operatorname{Aut} \mathbb{F}(z_1, z_2)$. Elements $R = \sum \alpha_i(z_1, z_2)\sigma_i$ of the group algebra $\mathbb{F}(z_1, z_2)[\Gamma]$ act as linear operators on $\mathbb{F}(z_1, z_2)$ and we may define in this way R_{ij} as linear operators on $\mathbb{F}(z_1, z_2, z_3)$. Thus we may look for solutions of the Yang-Baxter equation $R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$ amongst such operators. Denote by P the operator $P \cdot f(z_1, z_2) = f(z_2, z_1)$.

Theorem 1.1. *The operator*

$$R = -\frac{\kappa}{z_1 - z_2} P + \left(1 + \frac{\kappa}{z_1 - z_2}\right) I = I + \frac{\kappa}{z_1 - z_2}(I - P)$$

satisfies the Yang-Baxter equation for any $\kappa \in \mathbb{F}$.

Proof. Consider an operator of the general form

$$R = \alpha(z_1 - z_2)P + \beta(z_1 - z_2)I$$

Then it is easily seen that R satisfies the Yang-Baxter equation if and only if

$$\alpha(x)\alpha(y) = \alpha(x - y)\alpha(y) + \alpha(x)\alpha(y - x)$$

and

$$\alpha(x)\alpha(y)^2 + \beta(y)\beta(-y)\alpha(x + y) = \alpha(x)^2\alpha(y) + \beta(x)\beta(-x)\alpha(x + y)$$

These equations are satisfied when $\alpha(x) = -\kappa/x$ and $\beta(x) = 1 - \alpha(x)$. Moreover these are essentially the only such solutions [5]. $\square$

In fact, (at least when $\mathbb{F}$ is the field of complex numbers) this operator is the limit as the spectral parameter tends to infinity of certain solutions of the Yang-Baxter equation with spectral parameter on meromorphic functions constructed by Shibukawa and Ueno. Recall that in [17], they showed that operators of the form

$$R(\lambda) = G(z_1 - z_2, \lambda)P - G(z_1 - z_2, \kappa)I$$

satisfied the Yang-Baxter equation

$$R_{12}(\lambda_1)R_{13}(\lambda_1 + \lambda_2)R_{23}(\lambda_2) = R_{23}(\lambda_2)R_{13}(\lambda_1 + \lambda_2)R_{12}(\lambda_1)$$

for any $\kappa \in \mathbb{F}$ if G was of the form

$$G(z, \lambda) = \frac{\theta'(0)\theta(\lambda + z)}{\theta(\lambda)\theta(z)}$$

and θ satisfied the equation

$$\theta(x+y)\theta(x-y)\theta(z+w)\theta(z-w)+\theta(x+z)\theta(x-z)\theta(w+y)\theta(w-y) \\ +\theta(x+w)\theta(x-w)\theta(y+z)\theta(y-z)=0$$

The principal solution of this equation is $\theta(z)=\theta_1(z)$, the usual theta function (as defined in, say, [18]), along with the degenerations of the theta functions, $\sin(z)$ and z, as one or both of the periods tend to infinity. Felder and Pasquier [11] showed that in the case where θ is a true theta function, these operators, when twisted and restricted to suitable subspaces, yield the Belavin R-matrices. When θ is trigonometric, the operator yields in a similar way the affinizations of the Cremmer-Gervais R-matrices [5]. Letting the spectral parameter tend to infinity in a suitable way yields a constant solution of the YBE on the function field which again yields the usual Cremmer-Gervais R-matrices on restriction to finite dimensional subspaces. In the rational case, the same twisting and restriction procedure yields the desired quantization of $r_{\mathfrak{p}}$.

When $\theta(z)=z$ we have $G(z,\lambda)=1/\lambda+1/z$. Sending λ to infinity (and adjusting by a factor of $-\kappa$), we obtain the solution of the Yang-Baxter equation given in the theorem above. Write $R=I+\kappa r$ where $r=(I-P)/(z_1-z_2)$. Then r is a particularly interesting operator. It satisfies the classical Yang-Baxter equation, both forms of the quantum Yang-Baxter equation and has square zero. Its quantization is then just the exponential $\exp\kappa r=I+\kappa r=R$.

Let V_n be the space of polynomials in z_1 of degree less than n. Then we may identify the space $V_n\otimes V_n$ with the subspace of $\mathbb{F}(z_1,z_2)$ consisting of polynomials of degree less than n in both z_1 and z_2. Since $R\cdot z_1^iz_2^j=z_1^iz_2^j+\kappa(z_1^iz_2^j-z_2^iz_1^j)/(z_1-z_2)$, R restricts to an operator on $V_n\otimes V_n$. With respect to the natural basis, R has the form

$$R(e_i\otimes e_j)=e_i\otimes e_j-\kappa\sum_k\eta(i,j,k)e_k\otimes e_{i+j-k-1}$$

where

$$\eta(i,j,k)=\begin{cases}1 & \text{if } i\le k<j\\ -1 & \text{if } j\le k<i\\ 0 & \text{otherwise}\end{cases}$$

We now apply a simple twist. Define the operator $\tilde{F}_p$ by $\tilde{F}_p\cdot f(z_1,z_2)=f(z_1+p,z_2-p)$.

Lemma 1.2. *Let $F=\tilde{F}_p$. Then F and the above R satisfy:*

1. $F_{21}=F_{12}^{-1}$
2. $F_{12}F_{13}F_{23}=F_{23}F_{13}F_{12}$
3. $R_{12}F_{23}F_{13}=F_{13}F_{23}R_{12}$
4. $R_{23}F_{12}F_{13}=F_{13}F_{12}R_{23}$

Hence $R_F=F_{21}^{-1}RF_{12}$ also satisfies the Yang-Baxter equation.

Proof. The four relations are routine verifications. The fact that R_F then satisfies the Yang-Baxter equation is a well-known fact about R-matrices extended to this slightly more general situation. □

Notice that $F_{21}^{-1}PF_{12}=P$ and $F_{21}^{-1}F_{12}=F^2=\tilde{F}_{2p}$. Taking $p=h/2$ yields

$$R_F=\tilde{F}_h+\frac{\kappa}{z_1-z_2+h}(\tilde{F}_h-P)$$

Notice that

$$R_F \cdot z_1^i z_2^j = (z_1+h)^i (z_2-h)^j + \kappa \frac{(z_1+h)^i(z_2-h)^j - z_2^i z_1^j}{z_1 - z_2 + h}$$

and again R_F restricts to an operator on $V_n \otimes V_n$.

Definition 1.3. Let n be a positive integer. Define

$$R_{\mathfrak{p}} = \tilde{F}_h - \frac{hn}{z_1 - z_2 + h}(\tilde{F}_h - P)$$

restricted to $V_n \otimes V_n$.

Putting all the above together yields the main result.

Theorem 1.4. *For any $h \in \mathbb{F}$ and positive integer n, $R_{\mathfrak{p}}$ satisfies the Yang-Baxter equation.*

1.2. **Explicit form of $R_{\mathfrak{p}}$.** We now find an explicit formula for the matrix coefficients of $R_{\mathfrak{p}}$ with respect to the natural basis.

Define the coefficients of $R_{\mathfrak{p}}$ by $R_{\mathfrak{p}} \cdot z_1^i z_2^j = \sum_{a,b} R_{ij}^{ab} z_1^a z_2^b$.

Proposition 1.5. *The coefficients of $R_{\mathfrak{p}}$ are given by*

$$R_{ij}^{ab} = (-1)^{j-b}\left[\binom{i}{a}\binom{j}{b} + n\sum_k (-1)^{k-a}\binom{i}{k}\binom{j+k-a-1}{b}\eta(j,k,a)\right] h^{i+j-a-b}$$

Proof. Recall that

$$R_{\mathfrak{p}} \cdot z_1^i z_2^j = (z_1+h)^i (z_2-h)^j - hn \frac{(z_1+h)^i(z_2-h)^j - z_2^i z_1^j}{z_1 - z_2 + h}$$

For the second term we note that

$$\frac{z_1^j z_2^i - (z_1+h)^i (z_2-h)^j}{z_1 - z_2 + h} = \sum_{k,b,a} (-1)^{j+k-a-b}\binom{i}{k}\binom{j+k-a-1}{b}\eta(j,k,a) h^{i+j-a-b-1} z_1^a z_2^b$$

Combining this with the binomial expansion of the first term yields the assertion. ∎

The explicit form of this matrix in the case when $n = 3$ can be found in [13, Page 136].

1.3. **The semiclassical limit.** The operator $R_{\mathfrak{p}}$ is a polynomial function of the parameter h of the form $I + rh + O(h^2)$. By working over a suitably extended field, we may assume that h is a formal parameter. Hence r satisfies the classical Yang-Baxter equation. We now verify that r is the boundary solution $r_{\mathfrak{p}}$ associated to the classical Cremmer-Gervais r-matrix found by Gerstenhaber and Giaquinto in [12].

Recall that their solution of the CYBE on the boundary of the component containing the modified Cremmer-Gervais r-matrix was (up to a scalar)

$$b_{CG} = n \sum_{i<j} \sum_{k=1}^{j-i} E_{i,j-k+1} \wedge E_{j,i+k} + \sum_{i,j} (n-j) E_{i,i} \wedge E_{j,j+1}.$$

(Here as usual we are taking the E_{ij} to be the basis of $\operatorname{End} V$ defined by $E_{ij}e_k = \delta_{jk}e_i$ for a fixed basis $\{e_1, \dots, e_n\}$ of V; we shall use the convention $x \wedge y = x \otimes y - y \otimes x$). To pass from the b_{CG} to our matrix $r_{\mathfrak{p}}$, one applies the automorphism $\phi(E_{ij}) = -E_{n+1-j,n+1-i}$. Thus our matrix is again a boundary solution but for a Cremmer-Gervais r-matrix associated to a different choice of parabolic subalgebras.

Theorem 1.6. *The operator $R_{\mathfrak{p}}$ is of the form $I + r_{\mathfrak{p}}h + O(h^2)$ where*

$$r_{\mathfrak{p}} \cdot z_1^i z_2^j = n \sum \eta(i,j,k) z_1^k z_2^{i+j-k-1} + i z_1^{i-1} z_2^j - j z_1^i z_2^{j-1}$$

In particular the matrix representation of $r_{\mathfrak{p}}$ with respect to the usual basis is

$$n \sum_{i<j} \sum_{k=i}^{j-1} E_{k,i} \wedge E_{i+j-k-1,j} + \sum_{i,j} (j-1) E_{j-1,j} \wedge E_{i,i}.$$

Proof. From Proposition 1.5, the coefficients r_{ij}^{ab} are non-zero only when $b = i + j - a - 1$ and in this case,

$$\begin{aligned} r_{ij}^{a,i+j-a-1} &= \frac{1}{h} R_{ij}^{a,i+j-a-1} = (-1)^{a-i+1} \binom{i}{a} \binom{j}{a-i+1} + n\eta(i,j,a) \\ &= i\delta_{a,i-1} - j\delta_{a,i} + n\eta(i,j,a). \end{aligned}$$

Hence

$$r_{\mathfrak{p}} \cdot z_1^i z_2^j = n \sum \eta(i,j,k) z_1^k z_2^{i+j-k-1} + i z_1^{i-1} z_2^j - j z_1^i z_2^{j-1}.$$

Thus interpreting $r_{\mathfrak{p}}$ as an operator on $V \otimes V$ we get

$$r_{\mathfrak{p}} \cdot e_i \otimes e_j = n \sum \eta(i,j,k) e_k \otimes e_{i+j-k-1} + (i-1) e_{i-1} \otimes e_j - (j-1) e_i \otimes e_{j-1}.$$

In matrix form,

$$r_{\mathfrak{p}} = n \sum_{i<j} \sum_{k=i}^{j-1} E_{k,i} \wedge E_{i+j-k-1,j} + \sum_{i,j} (j-1) E_{j-1,j} \wedge E_{i,i}.$$

□

2. Boundary solutions of the Yang-Baxter equation

2.1. **The modified Yang-Baxter equation.** In [13], Gerstenhaber and Giaquinto introduced the *modified (quantum) Yang-Baxter equation* (MQYBE). An operator $R \in \operatorname{End} V \otimes V$ is said to satisfy the MQYBE if

$$R_{12}R_{13}R_{23} - R_{23}R_{13}R_{12} = \lambda(P_{123}R_{12} - P_{213}R_{23})$$

for some nonzero λ in $\mathbb{F}$. Here by P_{ijk} we mean the permutation operator $P_{ijk}(v_1 \otimes v_2 \otimes v_3) = v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes v_{\sigma(3)}$ where σ is the permutation (ijk).

Denote by $\mathfrak{R}$ the set of solutions of the YBE in $\operatorname{End} V \otimes V$ and by $\mathfrak{R}'$ the set of solutions of the MQYBE. Then $\mathfrak{R}'$ is a quasi-projective subvariety of $\mathbb{P}(M_{n^2}(\mathbb{F}))$ and $\bar{\mathfrak{R}}' - \mathfrak{R}'$ is contained in $\mathfrak{R}$ [13]. The elements of $\bar{\mathfrak{R}}' - \mathfrak{R}'$ are naturally called *boundary* solutions of the YBE. Little is currently known about this set though we conjecture that it contains some interesting R-matrices closely related to the quantizations of Belavin-Drinfeld r-matrices [9]. Let R be a solution of the YBE for which PR satisfies the Hecke equation $(PR - q)(PR + q^{-1}) = 0$. Set $\lambda = (1-q^2)^2/(1+q^2)^2$. Then $Q = (2R + (q^{-1} - q)P)/(q + q^{-1})$ is a unitary solution of the MQYBE. Roughly speaking what we expect to find is the following. If R is a quantization (in the algebraic sense) of a Belavin-Drinfeld r-matrix on $\mathfrak{sl}(n)$, then on the boundary

of the component of $\mathfrak{R}'$ containing Q, we should find the quantization of the skew-symmetric r-matrix associated (in the sense of Stolin) with the parabolic subalgebra of $\mathfrak{sl}(n)$ associated to r. We prove this conjecture here for the most well-known example, the Cremmer-Gervais R-matrices.

If $R \in \mathrm{End}(V \otimes V)\hat{\otimes}\mathbb{F}[[h]]$ satisfies the QYBE and is of the form $I + hr + O(h^2)$, then r satisfies the classical Yang-Baxter equation and R is said to be a quantization of r. The situation for the MQYBE is slightly more complicated and applies only to the $\mathfrak{sl}(n)$ case. Recall that the modified classical Yang-Baxter equation (MCYBE) for an element $r \in \mathfrak{sl}(n) \otimes \mathfrak{sl}(n)$ is the equation

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = \mu\Omega$$

where Ω is the unique invariant element of $\wedge^3\mathfrak{sl}(n)$ (which in the standard representation is the operator $P_{123} - P_{213}$). If R is of the form $I + hr + O(h^2)$ and is a solution of the MQYBE then λ is of the form $\nu h^2 + O(h^3)$ for some scalar ν. If $\nu \neq 0$, then r satisfies the MCYBE. In this case we say that R is a quantization of r.

There is an analogous notion of boundary solution for the classical Yang-Baxter equation. In [12], Gerstenhaber and Giaquinto showed that the matrix b_{CG} lies on the boundary of the component of the set of solutions to the MCYBE containing the modified Cremmer-Gervais classical r-matrix. They conjectured that its quantization should lie on the boundary of the component of $\mathfrak{R}'$ containing the modified Cremmer-Gervais R-matrix and proved this in the case $n = 3$ in [13]. We prove now this conjecture in general by showing that $R_{\mathfrak{p}}$ lies on the boundary of this component of $\mathfrak{R}'$.

2.2. **The Cremmer-Gervais solution of the MQYBE.** Consider the linear operator on $\mathbb{F}(z_1, z_2)$

$$R = \frac{\hat{q}pz_2}{pz_2 - z_1}P + \left(q - \frac{\hat{q}pz_2}{pz_2 - z_1}\right)F_p$$

where $\hat{q} = q - q^{-1}$ and $F_p \cdot f(z_1, z_2) = f(p^{-1}z_1, pz_2)$. When restricted to $V_n \otimes V_n$, the above operator becomes the usual 2-parameter Cremmer-Gervais R-matrix [5, 14]. When $p^n = q^2$, this is the original Cremmer-Gervais R-matrix which induces a quantization of $SL(n)$ [3].

If R is any solution of the YBE for which PR satisfies the Hecke equation $(PR - q)(PR + q^{-1}) = 0$ then $Q = (2R + (q^{-1} - q)P)/(q + q^{-1})$ is a unitary solution of the MQYBE for $\lambda = (1 - q^2)^2/(1 + q^2)^2$. Hence the operator $Q_{p,q} = (2R - \hat{q}P)/(q + q^{-1})$ satisfies the MQYBE. Explicitly,

$$Q_{p,q} = F_p - \frac{\hat{q}(z_2 + p^{-1}z_1)}{(q + q^{-1})(z_2 - p^{-1}z_1)}(F_p - P).$$

We call the corresponding matrices induced from these operators, the modified Cremmer-Gervais R-matrices.

2.3. **Deformation to the boundary.** Henceforth take $q^2 = p^n$. Then the operator $Q_{p,q}$ becomes

$$Q_p = F_p - \frac{(p^n - 1)(z_2 + p^{-1}z_1)}{(p^n + 1)(z_2 - p^{-1}z_1)}(F_p - P)$$

This is the modified version of the one-parameter Cremmer-Gervais operator described above. Again Q_p may be restricted to the subspace $V_n \otimes V_n$ where its action is given by

$$Q_p \cdot z_1^i z_2^j = p^{j-i} z_1^i z_2^j - \frac{(p^n-1)}{(p^n+1)} \sum [\eta(i,j,k) + \eta(i,j,k-1)] p^{j-k} z_1^k z_2^{i+j-k}.$$

Fix $h \in \mathbb{F}$ and $p \in \mathbb{F}^*$, define $\tilde{F}_{p,h}$ by $\tilde{F}_{p,h} \cdot f(z_1, z_2) = f(p^{-1}z_1 + p^{-1}h, pz_2 - h)$. Define further,

$$B_{p,h,n} = \tilde{F}_{p,h} - \frac{(p^n-1)(pz_2+z_1)}{(p^n+1)(pz_2-z_1-h)}(\tilde{F}_{p,h} - P) + \frac{h(p^n-1)(p+1)}{(p^n+1)(p-1)(pz_2-z_1-h)}(\tilde{F}_{p,h} - P)$$

Note that

$$B_{1,h,n} = \frac{hn}{(z_2 - z_1 - h)}(\tilde{F}_h - P) + \tilde{F}_h$$

since $\tilde{F}_h = \tilde{F}_{1,h}$. This is the operator R_F described above (with $\kappa = -hn$) that restricts to $R_{\mathfrak{p}}$ on finite dimensional subspaces.

Proposition 2.1. *For all h and $p \neq 1$, $B_{p,h,n}$ is a solution of the MQYBE similar to Q_p.*

Proof. Define a shift operator $\phi_t : \mathbb{F}(z_1, z_2) \to \mathbb{F}(z_1, z_2)$ by $\phi_t \cdot f(z_1, z_2) = f(z_1 - t, z_2 - t)$ and let ϕ_t act as usual on operators by conjugation. Then, if $F_{p,t} = \phi_t \circ F_p$,

$$\phi_t \circ Q_p = F_{p,t} - \frac{(p^n-1)(pz_2 + z_1 - t(p+1))}{(p^n+1)(pz_2 - z_1 - t(p-1))}(F_{p,t} - P)$$

Choose $t = h/(p-1)$. Then $\phi_t \circ Q_p = B_{p,h,n}$. This shows that $B_{p,h,n}$ is similar to Q_p and hence satisfies the MQYBE when $p \neq 1$. $\square$

Now the restriction of $B_{p,h,n}$ to $V_n \otimes V_n$ is a rational function of p which belongs to $\mathfrak{R}'$ and which for $p = 1$ is $R_{\mathfrak{p}}$. Thus $R_{\mathfrak{p}}$ must be a "boundary solution" of the Yang-Baxter equation.

3. Vertex-IRF transformations and solutions of the dynamical YBE

The original construction of the Cremmer-Gervais R-matrices was by a generalised kind of change of basis (a "vertex-IRF transformation") from the Gervais-Neveu solution of the constant dynamical Yang-Baxter equation. Given the above construction of $R_{\mathfrak{p}}$ as a rational degeneration of the Cremmer-Gervais matrices, it is natural to expect that $R_{\mathfrak{p}}$ should be connected in the same way with some kind of rational degeneration of the Gervais-Neveu matrices. In fact this is precisely what happens. The appropriate solutions to the constant dynamical Yang-Baxter equation (DYBE) were found by Etingof and Varchenko in [7]. In classifying certain kinds of solutions to the constant DYBE, they found that all such solutions were equivalent to either a generalized form of the Gervais-Neveu matrix or to a rational version of this matrix. It turns out that $R_{\mathfrak{p}}$ is connected via a vertex-IRF transformation with the simplest of this family of rational solutions to the constant DYBE.

Recall the framework for the dynamical Yang-Baxter equation given in [15]. Let H be a commutative cocommutative Hopf algebra. Let B be an H-module

algebra with structure map $\sigma : H \otimes B \to B$. Denote by $\mathcal{C}$ the category of right H-comodules. Define a new category $\mathcal{C}_\sigma$ whose objects are right H-comodules but whose morphisms are $\hom_{\mathcal{C}_\sigma}(V,W) = \hom_H(V, W \otimes B)$ where B is given a trivial comodule structure. Composition of morphisms is given by the natural embedding of $\hom_H(V, W\otimes B)$ inside $\hom_H(V\otimes B, W\otimes B)$.

A tensor product $\tilde{\otimes} : \mathcal{C}_\sigma \times \mathcal{C}_\sigma \to \mathcal{C}_\sigma$ is defined on this category in the following way. For objects V and W, $V\tilde{\otimes}W$ is the usual tensor product of H comodules $V\otimes W$. In order to define the tensor product of two morphisms, define first for any H-comodule W, a linear twist map $\tau : B\otimes W \to W \otimes B$ by

$$\tau(b\otimes w) = w_{(0)} \otimes \sigma(w_{(1)}\otimes b).$$

where $w \mapsto \sum w_{(0)} \otimes w_{(1)}$ is the structure map of the comodule W. Then for any pair of morphisms $f : V \to V'$ and $g : W \to W'$, define

$$f\tilde{\otimes}g = (1\otimes m_B)(1\otimes\tau\otimes 1)(f\otimes g)$$

Etingof and Varchenko showed in [7, 8] that the bifunctor $\tilde{\otimes}$ makes $\mathcal{C}_\sigma$ into a tensor category. Let $V \in \mathcal{C}_\sigma$ For any $R \in \mathrm{End}_{\mathcal{C}_\sigma}(V\tilde{\otimes}V)$ we define elements of $\mathrm{End}_{\mathcal{C}_\sigma}(V\tilde{\otimes}V\tilde{\otimes}V)$, $R_{12} = R\tilde{\otimes}1$ and $R_{23} = 1\tilde{\otimes}R$. Then R is said to satisfy the σ-dynamical braid equation (σ-DBE) if $R_{12}R_{23}R_{12} = R_{23}R_{12}R_{23}$. If R is a solution of the σ-DBE then RP satisfies the σ-dynamical Yang-Baxter equation:

$$R_{12}R_{23}^{12}R_{12}^{123} = R_{23}R_{12}^{23}R_{23}^{132}$$

where for instance $R_{12}^{132} = P_{132}R_{12}P_{123}$.

A vertex-IRF transformation of a solution of the σ-DBE can then be defined [15, Section 3.3] as an invertible linear operator $A : V \to V\otimes B$ (that is, invertible in the sense of the composition of such operators defined above) such that the conjugate operator $R^A = A_2^{-1}A_1^{-1}RA_1A_2$ is a “scalar” operator in the sense that $R^A(V\otimes V) \subset V\otimes V\otimes \mathbb{F}$. In this case R^A satisfies the traditional braid equation [15, Proposition 3.3]. Thus a vertex-IRF transformation transforms a solution of the σ-DYBE to a solution of the usual YBE.

Let T be the usual maximal torus of $SL(n)$. Let V be the standard representation of $SL(n)$ considered as a comodule over $H = \mathbb{F}[T]$ which we may consider as the group algebra of the weight lattice P; i.e., $H = \mathbb{F}[K_\lambda \mid \lambda \in P]$. Then V has a basis $\{e_i\}$ of weight vectors with weights ν_i. Denote the structure map by $\rho : V \to V\otimes\mathbb{F}[T]$. Then $\rho(e_i) = e_i \otimes K_{\nu_i}$.

Let $S(\mathfrak{h}^*)$ be the symmetric algebra on $\mathfrak{h}^*$ and set $B = \mathrm{Frac}(S(\mathfrak{h}^*))$. Define an action $\sigma : H\otimes B \to B$ by

$$\sigma(K_\lambda \otimes \nu) = \nu - (\lambda, \nu).$$

Denote $\sigma(K_\lambda\otimes b)$ by b^λ. Recall that $(\nu_i, \nu_j) = \delta_{ij} - 1/n$. This fact will be used repeatedly in the calculations below.

Let R be the matrix $R_\mathfrak{p}$ defined in Section 1.1 with $h = 1/n$, considered as an operator on the space $V\otimes V$ where V has basis $\{e_1, \ldots, e_n\}$. Set $\tilde{R} = RP$ and let $\tilde{R}_{ij}^{kl}$ be the matrix coefficients of $\tilde{R}$ defined by $\tilde{R}\cdot e_i\otimes e_j = \sum_{k,l}\tilde{R}_{ij}^{kl} e_k\otimes e_l$. From Definition 1.3 we have that for any z_1 and z_2,

$$\sum_{k,l}\tilde{R}_{ij}^{kl} z_1^{k-1}z_2^{l-1} = \alpha(z_1 - z_2)z_1^{i-1}z_2^{j-1} + \beta(z_1-z_2)(z_1+1/n)^{j-1}(z_2-1/n)^{i-1}.$$

where $\alpha(x) = 1/(x+1/n)$ and $\beta(x) = 1-\alpha(x)$. Define the operator $\mathcal{R} \in \mathrm{End}_{\mathcal{C}_\sigma} V\tilde{\otimes}V$ by

$$\begin{aligned}\mathcal{R}(e_i \otimes e_j) &= e_i \otimes e_j \otimes \alpha(\nu_i^{\nu_j} - \nu_j) + e_j \otimes e_i \otimes \beta(\nu_i^{\nu_j} - \nu_j)\\ &= e_i \otimes e_j \otimes \frac{1}{\nu_i - \nu_j + \delta_{ij}} + e_j \otimes e_i \otimes \left(1 - \frac{1}{\nu_i - \nu_j + \delta_{ij}}\right).\end{aligned}$$

This is the solution of the DBE corresponding to the standard example of solution of the DYBE of the type given in [7, Theorem 1.2]. Finally define an operator $A \in \mathrm{End}_{\mathcal{C}_\sigma}(V)$ by $A(e_i) = \sum e_k \otimes \nu_k^{i-1}$.

Theorem 3.1. $\mathcal{R}^A = \tilde{R}$

Proof. We prove that $\mathcal{R}A_1A_2 = A_1A_2\tilde{R}$. In matrix form this is equivalent to

$$\sum_{c,d} \mathcal{R}_{cd}^{ms}(A_i^c)^{\nu_d} A_j^d = \sum_{k,l} \tilde{R}_{ij}^{kl}(A_k^m)^{\nu_s} A_l^s.$$

Using the fact that $\beta(\nu_m^{\nu_s} - \nu_s) = 0$ when $m = s$

$$\begin{aligned}\sum_{k,l} \tilde{R}_{ij}^{kl}(A_k^m)^{\nu_s} A_l^s &= \sum_{k,l} \tilde{R}_{ij}^{kl}(\nu_m^{\nu_s})^{k-1}\nu_s^{l-1}\\ &= \alpha(\nu_m^{\nu_s} - \nu_s)(\nu_m^{\nu_s})^{i-1}\nu_s^{j-1} + \beta(\nu_m^{\nu_s} - \nu_s)(\nu_s - \frac{1}{n})^{i-1}(\nu_m^{\nu_s} + \frac{1}{n})^{j-1}\\ &= \alpha(\nu_m^{\nu_s} - \nu_s)(\nu_m^{\nu_s})^{i-1}\nu_s^{j-1} + \beta(\nu_m^{\nu_s} - \nu_s)(\nu_s^{\nu_m})^{i-1}(\nu_m)^{j-1}\\ &= \sum_{c,d} \mathcal{R}_{cd}^{ms}(A_i^c)^{\nu_d} A_j^d\end{aligned}$$

as required. □

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